Details of Mittag-Leffler random variate generation

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First type Mittag-Leffler distribution

Random variate generation

For the efficient generation of random variates, we use the following useful fact (see e.g. Theorem 19.1 in Haubold, Mathai, and Saxena (2011)): A standard α-Mittag-Leffler random variable Y has the representation:

$$Y \stackrel{d}{=} X^{1/\alpha} Z$$

where X is standard exponentially distributed, Z is α-stable with Laplace Transform

$$\mathbf{E}[\exp(-sZ)] = \exp(-s^\alpha),$$

X and Z are independent, and $\stackrel{d}{=}$ means equality in distribution.

```
n <- 5
x <- rexp(n)
```

Generating X

Generating Z To generate such random variates Z, we use

```
a <- 0.8
sigma <- (cos(pi*a/2))^(1/a)
z <- stabledist::rstable(n = n, alpha = a, beta = 1, gamma = sigma, delta = 0, pm = 1)
```

Below are the details of the calculation. We use the parametrization of the stable distribution by Samorodnitsky and Taqqu (1994) as it has become standard. For $\alpha \in (0,1)$ and $\alpha \in (1,2)$,

$$\mathbf{E}[\exp(itZ)] = \exp\left\{-\sigma^\alpha |t|^\alpha \left[1 - i\beta \operatorname{sgn} t \tan\frac{\pi\alpha}{2}\right] + iat\right\}$$

As in Meerschaert and Scheffler (2001), Equation (7.28), set

$$\sigma^\alpha = C\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2},$$

for some constant $C > 0$, set $\beta = 1$, set $a = 0$, and the log-characteristic function becomes

$$
\begin{aligned}
&-C\frac{\Gamma(2-\alpha)}{1-\alpha}\cos\frac{\pi\alpha}{2}|t|^{\alpha}\left[1-i\,\mathrm{sgn}(t)\tan\frac{\pi\alpha}{2}\right] && (1)\\
&=-C\Gamma(1-\alpha)|t|^{\alpha}\left[\cos\frac{\pi\alpha}{2}-i\,\mathrm{sgn}(t)\sin\frac{\pi\alpha}{2}\right] && (2)\\
&=-C\Gamma(1-\alpha)|t|^{\alpha}\left(\exp(-i\mathrm{sgn}(t)\pi/2)\right)^{\alpha} && (3)\\
&=-C\Gamma(1-\alpha)(-i|t|\mathrm{sgn}(t))^{\alpha} && (4)\\
&=-C\Gamma(1-\alpha)(-it)^{\alpha} && (5)
\end{aligned}
$$

Setting $t = is$ recovers the Laplace transform, and to match the Laplace transform $\exp(-s^{\alpha})$ of Z, it is necessary that $C\Gamma(1-\alpha) = 1$. But then $\sigma^{\alpha} = \cos(\pi\alpha/2)$, and we see that

$$Z \sim S(\alpha, \beta, \sigma, a) = S(\alpha, 1, \cos(\pi\alpha/2)^{1/\alpha}, 0)$$

```
y <- x^(1/a) * z
y
```

Generating Y

```
## [1] 0.1410581 0.3513453 0.1126169 0.6729680 0.7522310
```

References

Haubold, H. J., A. M. Mathai, and R. K. Saxena. 2011. "Mittag-Leffler Functions and Their Applications." *J. Appl. Math.* 2011: 1–51. https://doi.org/10.1155/2011/298628.

Meerschaert, Mark M, and Hans-Peter Scheffler. 2001. *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*. Book. First. New York: Wiley-Interscience.

Samorodnitsky, Gennady, and Murad S Taqqu. 1994. *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Stochastic Modeling. London: Chapman Hall.