UNIVERSITY OF BRISTOL

School of Mathematics

PROBABILITY 1

MATH 11300

(Paper code MATH-11300J)

January 2018   1 hour 30 minutes

This paper contains two sections: Section A and Section B.
Each section should be answered in a separate answer book.

Section A contains FIVE questions and Section B contains TWO questions.

All SEVEN answers will be used for assessment.

Calculators of an approved type (non-programmable, no text facility) are permitted.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

A1 (**8 marks**) Consider two well-shuffled piles of cards. The first pile contains five cards, marked with capital letters $\{V, W, X, Y, Z\}$ respectively, in some order. The second pile contains five cards, marked with small letters $\{v, w, x, y, z\}$, in some order. We randomly draw one card from each pile.

(a) How many elementary outcomes form the sample space?

(b) Let B be the event that we draw at least one of $\{v, V\}$. By listing and counting elementary outcomes, what is the probability $\mathbb{P}(B)$?

(c) Let C be the event that we draw $\{v\}$ and D be the event that we draw $\{V\}$. Write down $\mathbb{P}(C)$, $\mathbb{P}(D)$ and $\mathbb{P}(C \cap D)$. Explain how to use these three numbers to verify your value for $\mathbb{P}(B)$.

A2 (**8 marks**) Professor X has to give a lecture on three days out of seven each week. On days when he lectures, he has probability 0.9 of wearing a tie. On days when he does not lecture, he has probability 0.9 of not wearing a tie.

(a) By describing appropriate events and using the partition theorem, what is the probability that Professor X wears a tie on a randomly chosen day?

(b) Professor X is wearing a tie. What is the probability that he is giving a lecture today?

A3 (**8 marks**) A fidget spinner has three fixed arms, exactly one of which will point towards me after being spun around; each arm is equally likely to point towards me. We label the arms with the numbers 0, 1 and 2 respectively, and write X for the random variable giving the number written on the arm that points towards me.

Given that $X = x$, we toss a fair coin exactly x times, and write Y for the random variable for the total number of heads that we see (note that if $X = 0$ then $Y = 0$).

(a) Write down the joint probability mass function of X and Y.

(b) Find the marginal probability mass function of Y.

(c) Calculate the expectation $\mathbb{E}(Y)$.

A4 (**8 marks**) Random variable X has expectation 3 and variance 9. Random variable Y has expectation 3 and variance 16. X and Y are independent of one another.

Random variables U and V are defined by $U = 2X + Y$ and $V = X - 2Y$.

(a) Calculate the expectation and variance of U, and the expectation and variance of V.

(b) Calculate the covariance Cov (U, V). Are U and V independent?

A5 (**8 marks**) We write $Z \sim \mathcal{N}(\mu, \sigma^2)$ to denote a normal random variable having mean μ and variance σ^2. Suppose that $X \sim \mathcal{N}(4, 9)$, independent of $Y \sim \mathcal{N}(2, 16)$.

(a) Calculate the mean and standard deviation of $X + Y$.

(b) Use Chebyshev's inequality to give an upper bound on the probability $\mathbb{P}(X + Y \geq 14)$.

(c) Give the exact value of $\mathbb{P}(X + Y \geq 14)$, using the table presented on the last page.

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B1 In this question, we will consider the probability mass function of a Poisson random variable Z with parameter λ, which we will write as

$$p_\lambda(z) = \frac{e^{-\lambda}\lambda^z}{z!} \text{ for } z = 0, 1, \ldots.$$

Recall that $\mathbb{E}\left(g(Z)\right) = \sum_{z=0}^{\infty} p_\lambda(z) g(z)$, for any function g.

(a) i. (**3 marks**) Show that $\sum_{z=0}^{\infty} p_\lambda(z) = 1$, and express probabilities $\mathbb{P}(Z = 2)$ and $\mathbb{P}(Z \geq 1)$ as simply as possible.

ii. (**4 marks**) Calculate the expectations $\mathbb{E}(Z)$ and $\mathbb{E}\left(Z(Z-1)\right)$.

iii. (**3 marks**) Use your previous answers to calculate the value of $\mathbb{E}(Z^2)$ and hence find the variance Var (Z).

iv. (**5 marks**) Calculate the moment generating function $M_Z(t) = \mathbb{E}\left(e^{tZ}\right)$. By calculating appropriate derivatives of $M_Z(t)$ show how you can confirm the values of $\mathbb{E}(Z)$ and $\mathbb{E}(Z^2)$ calculated above.

(b) Anna is using Snapchat to communicate with her friend Becky. In a 5 minute period, the number of messages Z she sends to Becky has a Poisson distribution with parameter λ. However, Becky is distracted by Skyping her friend Caitlin, and so she only actually reads each message from Anna with probability p, independently of all other messages. We write Y for the number of messages that Becky reads.

i. (**3 marks**) Explain why the conditional probability $\mathbb{P}(Y = y|Z = z)$ has a binomial distribution for each value of z, and explicitly write down the formula for $\mathbb{P}(Y = y|Z = z)$.

ii. (**6 marks**) Write down the joint probability mass function of Y and Z, and calculate the marginal probability mass function for Y. State the distribution of Y, including its parameters. What is the probability that Becky reads no messages?
(Hint: you may find it helpful to use the fact that $Y \leq Z$).

iii. (**2 marks**) Define the conditional expectation $A(z) = \mathbb{E}[Y|Z = z]$, and write down an explicit formula for $A(z)$ in this case. Using the tower law in the form

$$\mathbb{E}(Y) = \mathbb{E}\left(\mathbb{E}[Y|Z]\right) = \mathbb{E}\left(A(Z)\right),$$

calculate the expected value of Y, and explain why this is consistent with the distribution of Y you found above.

iv. (**4 marks**) Using without proof the fact that the tower law implies

$$\mathbb{E}(YZ) = \mathbb{E}\left(ZA(Z)\right),$$

calculate the covariance of Y and Z. Explain why the sign of this covariance is not a surprise, and find the correlation coefficient of Y and Z.

B2 (a) Simon applies for a series of jobs until he is made an offer; he is offered the ith job with probability p, independently of all other job decisions.

i. (**4 marks**) Define the indicator random variable

$$U = \begin{cases} 1 & \text{if Simon is offered the first job he applies for,} \\ 0 & \text{if Simon is not offered the first job he applies for.} \end{cases}$$

Calculate the value of $\mathbb{E}(U^n)$ for each non-negative integer n, and deduce the variance of U.

ii. (**5 marks**) Write X for the random variable representing the number of the job that he is offered (so $X = i$ means that Simon is offered the ith job he applied for). Argue that X has a geometric distribution, and give the formula for the probability mass function of X. Calculate the probability $\mathbb{P}(X > x)$ for any integer $x = 1, 2, \ldots$.

iii. (**5 marks**) Given that Simon has not been offered the first 6 jobs that he applied for, calculate the probability that he will not have any job offers up to and including $6 + x$, for $x = 1, 2, \ldots$. Comment on what this shows.

(b) Consider random variable Z, which has an exponential distribution with parameter λ.

i. (**5 marks**) Calculate the distribution function $F_Z(z)$, the mean of Z and the probability $\mathbb{P}(a < Z \leq b)$ for any $0 \leq a < b$.

ii. (**5 marks**) Calculate the probability density function of Z^2, showing all your working.

iii. (**6 marks**) Define an integer-valued random variable $Y = \lceil Z \rceil$, where $\lceil z \rceil$ is the ceiling of z (the smallest integer greater than or equal to z).
Show that Y has a geometric probability mass function, with some parameter which you should specify. For what value of λ does Y have the same distribution as Simon's random variable X described above?

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$\Phi(z)$

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936

End of examination.