Bayesian Modelling B sheet 4

1. Recall from 1st year probability that
$$E[E(X|Y)] = E(X).$$
Apply this identity with X replaced by X^2 and simplify to prove that
$$E[\text{var}(X|Y)] + \text{var}[E(X|Y)] = \text{var}(X)$$
and hence that $E[\text{var}(X|Y)] \leq \text{var}(X)$.
By applying this result to the posterior distribution of a parameter θ and hyperparameter ϕ, conclude that *on average* (what does this mean, here?)
$$\text{var}(\theta|\phi, y) \leq \text{var}(\theta|y)$$
This provides support for the statements on slide 51.

2. Consider Monte Carlo integration of the function $g(u) = u^r$ for some fixed $r \geq 0$, over the interval $u \in [0, 1]$. Calculate the variances σ^2 and $\tilde{\sigma}^2$ associated with the crude and hit-or-miss Monte Carlo methods, for upper bounds $M = 1$ and $M = 10$. What happens if you use $M = 0.1$?

3. If you want to estimate π by the method suggested on slide 63, how many independent random points must be drawn from the square in order to get an approximate 95% confidence interval for π of length 0.1? What about 0.01 or 0.001? (Use the normal approximation to the binomial).

4. Suppose that $f_1(x)$ and $f_2(x)$ are two probability density functions, with $w(x) = f_1(x)/f_2(x) < \infty$ for all x. Let $g(x)$ be another function, whose expectation is required. By observing that
$$\int g(x)f_1(x)dx = \int [g(x)w(x)]f_2(x)dx$$
show that we can estimate $\int g(x)f_1(x)dx$ by simulating $x^{(1)}, x^{(2)}, \ldots$ independently from $f_2(x)$ and computing the weighted mean
$$\frac{1}{N}\sum_{t=1}^{N}[g(x^{(t)})w(x^{(t)})].$$
Show that this is unbiased, and find its variance. [This is the basis of the method of *importance sampling*.]

5. Let $f(x)$ be a probability density function for a scalar random variable. A point (X, Y) is drawn at random, uniformly in the area under the graph of the function $Mf(x)$, where M is an arbitrary positive constant. By considering the probability that $X \leq x$ for an arbitrary real x, show that the marginal distribution of X has density precisely $f(x)$.

6. Let $f_1(x)$ and $f_2(x)$ be two probability density functions, with $w(x) = f_1(x)/f_2(x) < \infty$ for all x. Let X have density function $f_2(x)$, and $U \sim \text{Uniform}(0, 1)$. Show that the conditional distribution of X given that $U < w(X)$ has density $f_1(x)$. [This is the basis of the method of *rejection sampling*.]